SOME PROPERTIES OF THE PSI AND POLYGAMMA FUNCTIONS

FENG QI AND BAI-NI GUO

Abstract. In this paper, some monotonicity and concavity results of several functions involving the psi and polygamma functions are proved, and then some known inequalities are extended and generalized.

1. Introduction

It is well-known that the classical Euler's gamma function $\Gamma(x)$ plays a central role in the theory of special functions and has much extensive applications in many branches, for example, statistics, physics, engineering, and other mathematical sciences. The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$, is called the psi or digamma function, and $\psi^{(i)}(x)$ for $i\in\mathbb{N}$ are known as the polygamma or multigamma functions.

In [5, Theorem 2], it was discovered that if $a\le-\ln 2$ and $b\ge 0$, then

$$a-\ln(e^{1/x}-1)<\psi(x)<b-\ln(e^{1/x}-1) \tag{1.1}$$

holds for $x>0$. In [4, Theorem 2.8], inequality (1.1) was sharpened: If $a\le-\gamma$ and $b\ge0$, then inequality (1.1) is valid for $x>0$, where the constants $-\gamma=-0.577\dots$, the negative of Euler-Mascheroni's constant, and 0 are the best possible.

The first aim of this paper is to generalize inequality (1.1) to an increasingly monotonic and concave properties as follows.

Theorem 1. *The function*

$$\phi(x)=\psi(x)+\ln\bigl(e^{1/x}-1\bigr) \tag{1.2}$$

is not only strictly increasing but also strictly concave on $(0,\infty)$, *with*

$$\lim_{x\to0^+}\phi(x)=-\gamma \quad \textit{and} \quad \lim_{x\to\infty}\phi(x)=0. \tag{1.3}$$

Remark 1. It is noted that the increasingly monotonic property in Theorem 1 was also obtained in [2] by using a different approach.

As direct consequences of the proof of Theorem 1, the following two inequalities for the trigamma function $\psi'(x)$ and the tetragamma function $\psi''(x)$ are deduced.

2000 *Mathematics Subject Classification.* Primary 33B15, 26A48, 26A51; Secondary 26D07.

Key words and phrases. psi function, polygamma function, monotonicity, convexity, concavity, inequality, necessary and sufficient condition, generalization, conjecture.

The first author was supported in part by the China Scholarship Council and the authors were supported in part by the NSF of Henan University, China.

This paper was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX.

Corollary 1. *For $x > 0$,*

$$\psi'(x) > \frac{e^{1/x}}{\left(e^{1/x}-1\right)x^2} \quad \textit{and} \quad \psi''(x) < \frac{e^{1/x}\left[1-2x\left(e^{1/x}-1\right)\right]}{\left(e^{1/x}-1\right)^2 x^4}. \tag{1.4}$$

The second aim of this paper is to extend Theorem 1 to the following necessary and sufficient conditions.

Theorem 2. *For $\theta > 0$, let*

$$\phi_\theta(x) = \psi(x) + \ln\left(e^{\theta/x}-1\right) \tag{1.5}$$

on $(0,\infty)$.

(1) *The function $\phi_\theta(x)$ is strictly increasing if and only if $0 < \theta \le 1$ and strictly decreasing if $\theta \ge 2$.*
(2) *The function $\phi_\theta(x)$ is strictly concave if $0 < \theta \le 1$ and strictly convex if $\theta \ge 2$.*
(3) $\lim_{x\to\infty} \phi_\theta(x) = \ln\theta$ *and*

$$\lim_{x\to0^+} \phi_\theta(x) = \begin{cases} -\gamma, & \theta = 1, \\ \infty, & \theta > 1, \\ -\infty, & 0 < \theta < 1. \end{cases} \tag{1.6}$$

As straightforward consequences of the proof of Theorem 2, the following inequalities for the trigamma function $\psi'(x)$ and the tetragamma function $\psi''(x)$ are presented, which extend the two inequalities in Corollary 1.

Corollary 2. *For $x > 0$, inequalities*

$$\psi'(x) > \frac{\theta e^{\theta/x}}{x^2\left(e^{\theta/x}-1\right)} \quad \textit{and} \quad \psi''(x) \le \frac{\theta e^{\theta/x}\left[\theta-2x\left(e^{\theta/x}-1\right)\right]}{x^4\left(e^{\theta/x}-1\right)^2} \tag{1.7}$$

hold if $0 < \theta \le 1$ and reverse if $\theta \ge 2$.

In [4, Theorem 2.6], inequality

$$-\gamma + x\psi'\left(\frac{x}{2}\right) < \psi(x+1) < -\gamma + x\psi'\left(\sqrt{x+1}-1\right) \tag{1.8}$$

for $x > 0$ was showed. Careful observation reveals that inequality (1.8) is not valid: If taking $x = 1$, the left-hand side inequality in (1.8) is reduced to $-\gamma + \frac{\pi^2}{2} < 1-\gamma$ which does not hold true clearly. After checking up the proof of [4, Theorem 2.6], it is found that inequality (1.8) can be corrected and extended as the following theorem.

Theorem 3. *If $x > 0$,*

$$-\gamma + x\psi'\left(1+\frac{x}{2}\right) < \psi(x+1) < -\gamma + x\psi'\left(\sqrt{x+1}\right). \tag{1.9}$$

If $-1 < x < 0$, inequality (1.9) is reversed.

As a generalization of Theorem 3, the following monotonicity are obtained.

Theorem 4. *The functions*

$$f(x) = \psi(x+1) - x\psi'\left(1+\frac{x}{2}\right) \quad \textit{and} \quad g(x) = x\psi'\left(\sqrt{x+1}\right) - \psi(x+1) \tag{1.10}$$

are both strictly increasing on $(-1,\infty)$*, with limits*

$$\lim_{x\to-1^+} f(x)=-\infty,\quad \lim_{x\to-1^+} g(x)=1+\gamma-\frac{\pi^2}{6},\quad \lim_{x\to\infty} f(x)=\lim_{x\to\infty} g(x)=\infty.$$

Remark 2. Making use of the difference equation (2.2) below, the functions $f(x)$ and $g(x)$ defined in (1.10) for $x\in(-1,\infty)$ can be rewritten as

$$f(x)=\begin{cases}\psi(x)-x\psi'\left(\dfrac{x}{2}\right)+\dfrac{5}{x}, & x\ne0\\ -\gamma, & x=0\end{cases}\tag{1.11}$$

and

$$g(x)=\begin{cases}x\psi'\bigl(\sqrt{x+1}\,\bigr)-\psi(x)-\dfrac{1}{x}, & x\ne0\\ \gamma, & x=0.\end{cases}\tag{1.12}$$

As an immediate consequence of the proof of Theorem 4, the monotonicity of the function

$$\bigl(u^2-1\bigr)\psi'(u)-\psi\bigl(u^2\bigr)$$

on $(-1,\infty)$ is derived as follows.

Theorem 5. *For* $x>-1$*, the function*

$$h(x)=\begin{cases}\bigl(x^2-1\bigr)\psi'(x)-\psi\bigl(x^2\bigr), & x\ne0\\ 1+\gamma-\dfrac{\pi^2}{6}, & x=0\end{cases}\tag{1.13}$$

is strictly increasing on $(-1,\infty)$*, with*

$$\lim_{x\to-1^+} h(x)=-\infty\quad\textit{and}\quad \lim_{x\to\infty} h(x)=\infty.\tag{1.14}$$

Remark 3. It is conjectured that the function $h(x)$ is strictly concave on $(-1,1)$ and strictly convex on $(1,\infty)$.

Finally, Theorem 4 can be generalized as follows.

Theorem 6. *If* i *is a positive odd, then the function*

$$f_i(x)=\psi^{(i)}(x+1)-x\psi^{(i+1)}\left(1+\frac{x}{2}\right)\tag{1.15}$$

is strictly decreasing on $(-1,\infty)$*; If* i *is a positive even, then the function* $f_i(x)$ *is strictly increasing on* $(-1,\infty)$*; For all* $i\in\mathbb{N}$*, the limits*

$$\lim_{x\to-1^+} f_i(x)=(-1)^{i+1}\infty\quad\textit{and}\quad \lim_{x\to\infty} f_i(x)=0\tag{1.16}$$

hold true.

Remark 4. Similar to the monotonic properties of the function $f_i(x)$, the following conjecture can be posed: If i is a positive odd, then the function

$$g_i(x)=\psi^{(i)}(x+1)-x\psi^{(i+1)}\bigl(\sqrt{x+1}\,\bigr)\tag{1.17}$$

is strictly decreasing on $(-1,\infty)$; If i is a positive even, then the function $g_i(x)$ is strictly increasing on $(-1,\infty)$; For all $i\in\mathbb{N}$, the limits

$$\lim_{x\to-1^+} g_i(x)=(-1)^{i+1}\infty\quad\text{and}\quad \lim_{x\to\infty} g_i(x)=0\tag{1.18}$$

are valid, except $\lim_{x\to\infty} g_1(x)=1$.

Direct calculation yields

$$\begin{aligned}[g_i(x)]' &= \psi^{(i+1)}(x+1) - \psi^{(i+1)}\big(\sqrt{x+1}\,\big) - \frac{x}{2\sqrt{x+1}}\psi^{(i+2)}\big(\sqrt{x+1}\,\big)\\ &= \frac{2u\big[\psi^{(i+1)}\big(u^2\big) - \psi^{(i+1)}(u)\big] - \big(u^2-1\big)\psi^{(i+2)}(u)}{2u}\\ &= \frac{\big[\psi^{(i)}\big(u^2\big) - \big(u^2-1\big)\psi^{(i+1)}(u)\big]'}{2u},\end{aligned}$$

where $u = \sqrt{x+1} > 0$ for $x > -1$. Therefore, in order to verify above conjecture, it is sufficient to show the monotonic properties of the function

$$\psi^{(i)}\big(u^2\big) - \big(u^2-1\big)\psi^{(i+1)}(u) \tag{1.19}$$

on $(0, \infty)$.

Remark 5. It is also natural to pose an open problem: For $i, k \in \mathbb{N}$ and positive numbers α, β, δ, λ, μ and τ, what about the monotonicities and convexities of the more general function

$$\varphi_{i,k}(x) = \psi^{(i-1)}(x+\alpha) - (x+\beta)^k\psi^{(i)}\big(\lambda(x+\delta)^\mu + \tau\big) \tag{1.20}$$

in an appropriate interval where it is defined?

2. Lemmas

The following lemmas are useful for the proofs of some of our theorems.

Lemma 1 ([9, Lemma 1])**.** *If $f(x)$ is a function defined in an infinite interval I such that*

$$f(x) - f(x+\varepsilon) > 0 \quad \text{and} \quad \lim_{x\to\infty} f(x) = \delta$$

for some $\varepsilon > 0$, then $f(x) > \delta$ on I.

Proof. By induction, for any $x \in I$,

$$f(x) > f(x+\varepsilon) > f(x+2\varepsilon) > \cdots > f(x+k\varepsilon) \to \delta$$

as $k \to \infty$. The proof of Lemma 1 is complete. □

Lemma 2 ([1])**.** *For $x > 0$ and $k \in \mathbb{N}$,*

$$\psi(x) - \ln x + \frac{1}{x} = \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t-1}\right)e^{-xt}\,\mathrm{d}\,t, \tag{2.1}$$

$$\psi^{(k-1)}(x+1) = \psi^{(k-1)}(x) + \frac{(-1)^{k-1}(k-1)!}{x^k}. \tag{2.2}$$

Recall [10, 16] that a function f is said to be completely monotonic on an interval I if it has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \ge 0 \tag{2.3}$$

for $x \in I$ and $n \ge 0$. The well-known Bernstein's Theorem [16, p. 161] states that a function f is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xs}\,\mathrm{d}\,\mu(s), \tag{2.4}$$

where μ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$. This expresses that a function f is completely monotonic on $(0, \infty)$ if and only if it is a Laplace transform of the measure μ.

Lemma 3 ([11, Theorem 2]). *The function*

$$\psi(x) - \ln x + \frac{\alpha}{x}$$

is completely monotonic on $(0, \infty)$ *if and only if* $\alpha \ge 1$ *and*

$$\ln x - \frac{\alpha}{x} - \psi(x)$$

is completely monotonic on $(0, \infty)$ *if and only if* $\alpha \le \frac{1}{2}$. *Consequently, inequalities*

$$\frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} < (-1)^{k+1}\psi^{(k)}(x) < \frac{(k-1)!}{x^k} + \frac{k!}{x^{k+1}} \tag{2.5}$$

hold for $x \in (0, \infty)$ *and* $k \in \mathbb{N}$.

Remark 6. Recall [3, 6, 8, 12, 13, 14, 15] that a positive function f on an interval I is called logarithmically completely monotonic if it satisfies

$$(-1)^k[\ln f(x)]^{(k)} \ge 0 \tag{2.6}$$

for $k \in \mathbb{N}$ on I. Lemma 3 can also be concluded from the necessary and sufficient conditions such that the function

$$\frac{e^x\Gamma(x)}{x^{x-\alpha}} \tag{2.7}$$

for $\alpha \in \mathbb{R}$ and its reciprocal are logarithmically completely monotonic on $(0, \infty)$, which was established in [7].

Lemma 4 ([1, pp. 259–260]). *For* $z \ne -1, -2, -3, \dots,$

$$\psi(1+z) = -\gamma + \sum_{n=1}^{\infty} \frac{z}{n(n+z)}. \tag{2.8}$$

For $n \in \mathbb{N}$ *and* $z \ne 0, -1, -2, \dots,$

$$\psi^{(n)}(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}}. \tag{2.9}$$

3. Proofs of theorems and corollaries

Now we are in a position to prove our theorems.

Proof of Theorem 1. Direct calculation gives

$$\phi'(x) = \psi'(x) - \frac{e^{1/x}}{\left(e^{1/x}-1\right)x^2}, \tag{3.1}$$

$$\phi''(x) = \psi''(x) + \frac{2e^{1/x}}{\left(e^{1/x}-1\right)x^3} + \frac{e^{1/x}}{\left(e^{1/x}-1\right)x^4} - \frac{e^{2/x}}{\left(e^{1/x}-1\right)^2x^4} \tag{3.2}$$

and

$$\lim_{x\to\infty}\phi'(x) = \lim_{x\to\infty}\phi''(x) = 0.$$

From the difference equation (2.2), it is deduced that

$$\begin{aligned} \phi'(x) - \phi'(x+1) &= \frac{1}{x^2} + \frac{1}{\left[1-1/e^{1/(x+1)}\right](x+1)^2} - \frac{1}{\left(1-1/e^{1/x}\right)x^2} \\ &= \frac{e^{1/(x+1)}}{[e^{1/(x+1)}-1](x+1)^2} - \frac{1}{\left(e^{1/x}-1\right)x^2}. \end{aligned} \tag{3.3}$$

It is easy to see that $\phi'(x)-\phi'(x+1)>0$ for $x>0$ is equivalent to

$$x^2\big(e^{1/x}-1\big)>(x+1)^2\big[1-e^{-1/(x+1)}\big]. \tag{3.4}$$

This can be expanded and simplified as

$$\sum_{k=3}^{\infty}\frac{1}{k!}\left[\frac{1}{x^{k-2}}+\frac{(-1)^k}{(x+1)^{k-2}}\right]>0$$

which is valid clearly. By Lemma 1, it is concluded that the function $\phi'(x)$ is positive and $\phi(x)$ is strictly increasing on $(0,\infty)$.

By utilization of (3.3) and differentiation, it is acquired directly that

$$\begin{aligned}\phi''(x)-\phi''(x+1)&=[\phi'(x)-\phi'(x+1)]'\\&=\frac{e^{1/(x+1)}\big[3+2x-2(x+1)e^{1/(x+1)}\big]}{\big[e^{1/(x+1)}-1\big]^2(x+1)^4}-\frac{(1-2x)e^{1/x}+2x}{\big(e^{1/x}-1\big)^2x^4}\end{aligned}$$

for $x>0$. It is obvious that the fact $\phi''(x)-\phi''(x+1)<0$ is equivalent to

$$\left[\frac{e^{1/x}-1}{e^{1/(x+1)}-1}\right]^2>\frac{1}{e^{1/(x+1)}}\left(\frac{x+1}{x}\right)^4\frac{e^{1/x}-2x\big(e^{1/x}-1\big)}{1-2(1+x)\big[e^{1/(x+1)}-1\big]}.$$

Considering (3.4), in order to prove above inequality, it is sufficient to show

$$\frac{1}{e^{1/(x+1)}}>\frac{e^{1/x}-2x\big(e^{1/x}-1\big)}{1-2(1+x)\big[e^{1/(x+1)}-1\big]}$$

which is equivalent to

$$3+2x-2e^{1/(x+1)}-(2x+1)e^{1/(x+1)+1/x}\triangleq h(x)<0.$$

Straightforward computation gives

$$h'(x)=\frac{2x^2e^{1/(x+1)}+2(x+1)^2x^2+\big(4x^2+4x+1-2x^4\big)e^{1/(x+1)+1/x}}{x^2(x+1)^2},$$

$$h''(x)=-\frac{2(2x+3)x^4+e^{1/x}\big(8x^5+26x^4+36x^3+24x^2+8x+1\big)}{x^4(x+1)^4e^{-1/(x+1)}}<0,$$

and $\lim_{x\to\infty}h'(x)=0$. Hence, the function $h'(x)$ for $x>0$ is decreasing and positive, and then the function $h(x)$ for $x>0$ is increasing. From $\lim_{x\to\infty}h(x)=-4$, it is deduced that $h(x)<-4<0$ for $x>0$. Consequently, utilizing Lemma 1, it is concluded that $\phi''(x)<0$ on $(0,\infty)$. The concavity of $\phi(x)$ is proved.

From (2.1), it follows that

$$\begin{aligned}\phi(x)&=\ln x-\frac{1}{x}+\ln\big(e^{1/x}-1\big)+\int_0^\infty\left(\frac{1}{t}-\frac{1}{e^t-1}\right)e^{-xt}\,\mathrm{d}\,t\\&=\ln\frac{\big(e^{1/x}-1\big)}{1/x}-\frac{1}{x}+\int_0^\infty\left(\frac{1}{t}-\frac{1}{e^t-1}\right)e^{-xt}\,\mathrm{d}\,t\\&\to 0\end{aligned}$$

as $x\to\infty$. Employing (2.2) for $i=1$ reveals

$$\begin{aligned}\phi(x)=\psi(x)+\frac{1}{x}+\ln\big(e^{1/x}-1\big)-\frac{1}{x}&=\psi(x+1)+\ln\big(e^{1/x}-1\big)-\ln e^{1/x}\\&=\psi(x+1)+\ln\big(1-e^{-1/x}\big)\to\psi(1)=-\gamma\end{aligned}$$

as $x\to0^+$. The proof of Theorem 1 is complete. □

Proof of Corollary 1. These inequalities follow from the increasing monotonicity and concavity of $\phi(x)$ and formulas (3.1) and (3.2). $\square$

Proof of Theorem 2. Using (2.1), the function $\phi_\theta(x)$ becomes

$$\begin{aligned}
\phi_\theta(x) &= \ln x - \frac{1}{x} + \ln\bigl(e^{\theta/x}-1\bigr) + \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t-1}\right) e^{-xt}\,\mathrm{d}\,t \\
&= \ln \frac{\bigl(e^{\theta/x}-1\bigr)}{1/x} - \frac{1}{x} + \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t-1}\right) e^{-xt}\,\mathrm{d}\,t \\
&\to \ln\theta
\end{aligned}$$

as $x\to\infty$. Employing (2.2) for $i=1$ reveals

$$\begin{aligned}
\phi_\theta(x) &= \psi(x) + \frac{1}{x} + \ln\bigl(e^{\theta/x}-1\bigr) - \frac{1}{x} \\
&= \psi(x+1) + \ln\bigl(e^{\theta/x}-1\bigr) - \ln e^{1/x} \\
&= \psi(x+1) + \ln\bigl(e^{(\theta-1)/x} - e^{-1/x}\bigr) \\
&\to \begin{cases} \psi(1) = -\gamma, & \theta = 1 \\ \infty, & \theta > 1 \\ -\infty, & 0<\theta<1 \end{cases}
\end{aligned}$$

as $x\to 0^+$. The two limits in Theorem 2 is proved.

Easy calculation yields

$$\phi_\theta'(x) = \psi'(x) - \frac{\theta e^{\theta/x}}{x^2\bigl(e^{\theta/x}-1\bigr)} \triangleq \psi'(x) - \varphi(\theta,x), \tag{3.5}$$

$$\phi_\theta''(x) = \psi''(x) + \frac{\theta e^{\theta/x}\bigl[2x\bigl(e^{\theta/x}-1\bigr)-\theta\bigr]}{x^4\bigl(e^{\theta/x}-1\bigr)^2} = \psi''(x) - \frac{\mathrm{d}\,\varphi(\theta,x)}{\mathrm{d}\,x}. \tag{3.6}$$

It is easy to verify that

$$\frac{\mathrm{d}\,\varphi(\theta,x)}{\mathrm{d}\,\theta} = \frac{e^{\theta/x}\bigl(e^{\theta/x}-1-\theta/x\bigr)}{x^2\bigl(e^{\theta/x}-1\bigr)^2} > 0$$

and

$$\begin{aligned}
\frac{\mathrm{d}^2\varphi(\theta,x)}{\mathrm{d}\,\theta\,\mathrm{d}\,x} &= -\frac{e^{\theta/x}\Bigl[(\theta/x)^2\bigl(e^{\theta/x}+1\bigr) - 4(\theta/x)\bigl(e^{\theta/x}-1\bigr) + 2\bigl(e^{\theta/x}-1\bigr)^2\Bigr]}{x^3\bigl(e^{\theta/x}-1\bigr)^3} \\
&= -\frac{2\theta^2 e^{\theta/x}\Bigl\{\bigl(e^{\theta/x}-1\bigr)/2 + \bigl[\bigl(e^{\theta/x}-1\bigr)/(\theta/x)-1\bigr]^2\Bigr\}}{x^5\bigl(e^{\theta/x}-1\bigr)^3} \\
&< 0
\end{aligned}$$

for $\theta>0$ and $x>0$. These implies that the functions $\varphi(\theta,x)$ and $\frac{\mathrm{d}\,\varphi(\theta,x)}{\mathrm{d}\,x}$ are increasing and decreasing with $\theta>0$ respectively. Consequently, by using the two inequalities in Corollary 1, it is concluded for $0<\theta\le 1$ and $x\in(0,\infty)$ that

$$\varphi(\theta,x) \le \varphi(1,x) = \frac{e^{1/x}}{x^2\bigl(e^{1/x}-1\bigr)} < \psi'(x)$$

and

$$\frac{\mathrm{d}\varphi(\theta,x)}{\mathrm{d}x} \ge \frac{\mathrm{d}\varphi(1,x)}{\mathrm{d}x} = \frac{e^{1/x}\left[1-2x\left(e^{1/x}-1\right)\right]}{\left(e^{1/x}-1\right)^2x^4} > \psi''(x).$$

Hence, the function $\phi_\theta'(x)$ is positive and $\phi_\theta''(x) < 0$ on $(0,\infty)$ for $0 < \theta \le 1$, and the function $\phi_\theta(x)$ is increasing and concave on $(0,\infty)$ for $0 < \theta \le 1$.

In order that $\phi_\theta'(x) < 0$, by the right-hand side inequality in (2.5), it is sufficient to prove

$$\frac{1}{x} + \frac{1}{x^2} - \frac{\theta e^{\theta/x}}{x^2\left(e^{\theta/x}-1\right)} \le 0$$

which is equivalent to

$$1 + \frac{u}{\theta} - \frac{ue^u}{e^u-1} \le 0$$

for $u = \frac{\theta}{x} > 0$. Therefore, it suffices to let

$$\theta \ge \frac{u(e^u-1)}{1+(u-1)e^u} \triangleq \delta(u)$$

for $u > 0$. Since $\delta(u)$ is decreasing and $1 < \delta(u) < 2$ on $(0,\infty)$, when $\theta \ge 2$, the function $\phi_\theta(x)$ is decreasing on $(0,\infty)$.

In order that $\phi_\theta''(x) > 0$, by the right-hand side inequality in (2.5), it is sufficient to show

$$\frac{\theta e^{\theta/x}\left[2x\left(e^{\theta/x}-1\right)-\theta\right]}{x^4\left(e^{\theta/x}-1\right)^2} \ge \frac{1}{x^2} + \frac{2}{x^3}$$

which is equivalent to

$$\frac{2ue^u(e^u-1-u/2)}{(e^u-1)^2} \ge 1 + \frac{2u}{\theta}$$

for $u = \frac{\theta}{x} > 0$. Therefore, it suffices to let

$$\frac{2}{\theta} \le \frac{1}{u}\left[\frac{2ue^u(e^u-1-u/2)}{(e^u-1)^2} - 1\right] \triangleq \rho(u)$$

for $u > 0$. Since $\rho(u)$ is increasing and $1 < \rho(u) < 2$ on $(0,\infty)$, it is sufficient to let $\theta \ge 2$.

If $\phi_\theta(x)$ is increasing on $(0,\infty)$, then $\phi_\theta'(x) > 0$ means that

$$[x^2\psi'(x) - \theta]e^{\theta/x} > x^2\psi'(x).$$

It is well-known that $\psi'(x) > 0$ on $(0,\infty)$, thus it is necessary that $\theta < x^2\psi'(x)$. Lemma 3 for $k = 1$ gives

$$\psi'(x) < \frac{1}{x} + \frac{1}{x^2}$$

on $(0,\infty)$, and then $\theta < x+1$ on $(0,\infty)$. Hence, the required necessary condition $\theta \le 1$ is proved. □

Proof of Corollary 2. These inequalities follow directly from the monotonicity and convexity of $\phi_\theta(x)$ and formulas (3.5) and (3.6). □

Proof of Theorem 3. By (2.8), it follows that

$$\psi(x+1) = -\gamma + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+x}\right) \tag{3.7}$$

for $x > -1$. By the mean value theorem for differentiation, it is obvious that there exists a number $\mu = \mu(k) = \mu(k, x)$ for $k \in \mathbb{N}$ such that $-1 < \mu(k) < x$ and

$$\frac{1}{k} - \frac{1}{k+x} = \frac{x}{[k+\mu(k)]^2}. \tag{3.8}$$

Employing (3.8) in (3.7) leads to

$$\psi(x+1) = -\gamma + x\sum_{k=1}^{\infty} \frac{1}{[k+\mu(k)]^2} \tag{3.9}$$

for $x > -1$. From (3.8), it is deduced that

$$\mu(k) = \sqrt{k(k+x)} - k.$$

It is not difficult to show that the mapping $k \to \mu(k)$ is strictly increasing on $[1, \infty)$ with

$$\mu(1) = \sqrt{1+x} - 1 > -1 \quad \text{and} \quad \lim_{k\to\infty} \mu(k) = \frac{x}{2}.$$

Hence, from (3.9) and (2.9), it is concluded that inequality

$$\begin{aligned} x\psi'\left(1+\frac{x}{2}\right) &= x\psi'(1+\lim_{k\to\infty}\mu(k)) = x\sum_{k=1}^{\infty}\frac{1}{[k+\lim_{k\to\infty}\mu(k)]^2} \\ &< \gamma + \psi(x+1) < x\sum_{k=1}^{\infty}\frac{1}{[k+\mu(1)]^2} = x\psi'(1+\mu(1)) = x\psi'\big(\sqrt{x+1}\big) \end{aligned}$$

holds for $x > 0$ and reverses for $-1 < x < 0$. The proof of Theorem 3 is finished. □

Proof of Theorem 4. Direct computation and utilization of the mean value theorem for differentiation gives

$$\begin{aligned} f'(x) &= \psi'(x+1) - \psi'\left(1+\frac{x}{2}\right) - \frac{x}{2}\psi''\left(1+\frac{x}{2}\right) \\ &= \frac{x}{2}\psi''(1+\xi(x)) - \frac{x}{2}\psi''\left(1+\frac{x}{2}\right) \\ &= \frac{x}{2}\left[\psi''(1+\xi(x)) - \psi''\left(1+\frac{x}{2}\right)\right], \end{aligned}$$

where $\xi(x)$ is between $\frac{x}{2}$ and x for $x > -1$. Since $\psi''(x)$ is strictly increasing on $(0, \infty)$, it follows clearly that $f'(x) > 0$ for $x \neq 0$. Hence, the function $f(x)$ is strictly increasing on $(-1, \infty)$.

Standard argument leads to

$$\begin{aligned} g'(x) &= \psi'\big(\sqrt{x+1}\big) - \psi'(x+1) + \frac{x}{2\sqrt{x+1}}\psi''\big(\sqrt{x+1}\big) \\ &= \frac{2u\big[\psi'(u) - \psi'\big(u^2\big)\big] + \big(u^2-1\big)\psi''(u)}{2u} \\ &= \frac{\big[\big(u^2-1\big)\psi'(u) - \psi\big(u^2\big)\big]'}{2u} \end{aligned}$$

for $x > -1$ and $u = \sqrt{x+1} > 0$. Utilization of formulas (2.8) and (2.9) and direct differentiation gives

$$\big(u^2-1\big)\psi'(u) - \psi\big(u^2\big) = \big(u^2-1\big)\sum_{i=0}^{\infty}\frac{1}{(u+i)^2} + \gamma - \sum_{i=0}^{\infty}\left(\frac{1}{1+i} - \frac{1}{u^2+i}\right)$$

$$= \gamma + \left(u^2 - 1\right) \sum_{i=0}^{\infty} \left[\frac{1}{(u+i)^2} - \frac{1}{(i+1)(u^2+i)} \right]$$

$$= \gamma + \sum_{i=1}^{\infty} \frac{i(u^2-1)(u-1)^2}{(i+1)(u+i)^2(u^2+i)} \tag{3.10}$$

and

$$\left[\left(u^2 - 1\right)\psi'(u) - \psi\left(u^2\right)\right]' = \sum_{i=1}^{\infty} \frac{2i(u-1)^2\left(u^3 + 2u^2 + 2iu + i\right)}{(i+u)^3\left(u^2+i\right)^2} > 0 \tag{3.11}$$

for $u > 0$. Hence, the function $g(x)$ is strictly increasing on $(-1, \infty)$.

It is apparent that

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \psi(x+1) + \psi'\left(\frac{1}{2}\right) = -\infty.$$

By (2.5) for $k = 1$ and $\lim_{x\to\infty} \psi(x) = \infty$, it is clear that $\lim_{x\to\infty} f(x) = \infty$.

By using (3.10), it is easy to see that

$$\lim_{x \to -1^+} g(x) = \lim_{u \to 0^+} \left[\left(u^2 - 1\right)\psi'(u) - \psi\left(u^2\right)\right]$$

$$= \gamma + \lim_{u \to 0^+} \sum_{i=1}^{\infty} \frac{i(u^2-1)(u-1)^2}{(i+1)(u+i)^2(u^2+i)} = \gamma - \sum_{i=1}^{\infty} \frac{1}{(i+1)i^2} = 1 + \gamma - \frac{\pi^2}{6}$$

and

$$\lim_{x \to \infty} g(x) = \lim_{u \to \infty} \left[\left(u^2 - 1\right)\psi'(u) - \psi\left(u^2\right)\right]$$

$$= \gamma + \lim_{u \to \infty} \sum_{i=1}^{\infty} \frac{i(u^2-1)(u-1)^2}{(i+1)(u+i)^2(u^2+i)} = \gamma + \sum_{i=1}^{\infty} \frac{i}{i+1} = \infty.$$

The proof of Theorem 4 is complete. □

Proof of Theorem 5. It is not difficult to see that the factor $u^3 + 2u^2 + 2iu + i$ for $i \in \mathbb{N}$ in formula (3.11) is positive if and only if $u > -1$. Therefore, the function $h(x)$ is strictly increasing on $(-1, \infty)$.

The limits can be derived from (2.8) and (2.9). □

Proof of Theorem 6. It is obvious that

$$[f_i(x)]' = \psi^{(i+1)}(x+1) - \psi^{(i+1)}\left(1 + \frac{x}{2}\right) - \frac{x}{2}\psi^{(i+2)}\left(1 + \frac{x}{2}\right)$$

$$= \frac{x}{2}\left[\psi^{(i+2)}(1 + \eta(x)) - \psi^{(i+2)}\left(1 + \frac{x}{2}\right)\right],$$

where $\eta(x)$ is between $\frac{x}{2}$ and x. This means $(-1)^i[f_i(x)]' \ge 0$ for $i \in \mathbb{N}$, and then the monotonicities of $f_i(x)$ on $(-1, \infty)$ for $i \in \mathbb{N}$ are proved.

The two limits can be deduced easily from inequality (2.5). □

References

[1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 4th printing, with corrections, Washington, 1965.

[2] H. Alzer, *Sharp inequalities for the harmonic numbers*, Expo. Math. **24** (2006), no. 4, 385–388.

[3] R. D. Atanassov and U. V. Tsoukrovski, *Some properties of a class of logarithmically completely monotonic functions*, C. R. Acad. Bulgare Sci. **41** (1988), no. 2, 21–23.

[4] N. Batir, *On some properties of digamma and polygamma functions*, J. Math. Anal. Appl. **328** (2007), no. 1, 452–465.

[5] N. Batir, *Some new inequalities for gamma and polygamma functions*, J. Inequal. Pure Appl. Math. **6** (2005), no. 4, Art. 103; Available online at http://jipam.vu.edu.au/article.php?sid=577.

[6] C. Berg, *Integral representation of some functions related to the gamma function*, Mediterr. J. Math. **1** (2004), no. 4, 433–439.

[7] Ch.-P. Chen and F. Qi, *Logarithmically completely monotonic functions relating to the gamma function*, J. Math. Anal. Appl. **321** (2006), no. 1, 405–411.

[8] A. Z. Grinshpan and M. E. H. Ismail, *Completely monotonic functions involving the gamma and q-gamma functions*, Proc. Amer. Math. Soc. **134** (2006), 1153–1160.

[9] F. Qi, *A completely monotonic function involving divided difference of psi function and an equivalent inequality involving sums*, ANZIAM J. **48** (2007), no. 4, 523–532.

[10] F. Qi, *Certain logarithmically N-alternating monotonic functions involving gamma and q-gamma functions*, Nonlinear Funct. Anal. Appl. **12** (2007), no. 4, 675–685.

[11] F. Qi, *Three classes of logarithmically completely monotonic functions involving gamma and psi functions*, Integral Transforms Spec. Funct. **18** (2007), no. 7, 503–509.

[12] F. Qi and Ch.-P. Chen, *A complete monotonicity property of the gamma function*, J. Math. Anal. Appl. **296** (2004), no. 2, 603–607.

[13] F. Qi and B.-N. Guo, *Complete monotonicities of functions involving the gamma and digamma functions*, RGMIA Res. Rep. Coll. **7** (2004), no. 1, Art. 8, 63–72; Available online at http://www.staff.vu.edu.au/rgmia/v7n1.asp.

[14] F. Qi, B.-N. Guo and Ch.-P. Chen, *Some completely monotonic functions involving the gamma and polygamma functions*, J. Aust. Math. Soc. **80** (2006), 81–88.

[15] F. Qi, B.-N. Guo and Ch.-P. Chen, *Some completely monotonic functions involving the gamma and polygamma functions*, RGMIA Res. Rep. Coll. **7** (2004), no. 1, Art. 5, 31–36; Available online at http://www.staff.vu.edu.au/rgmia/v7n1.asp.

[16] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1941.

(F. Qi) Research Institute of Mathematical Inequality Theory, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

E-mail address: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com

URL: http://qifeng618.spaces.live.com

(B.-N. Guo) School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

E-mail address: bai.ni.guo@gmail.com, bai.ni.guo@hotmail.com

URL: http://guobaini.spaces.live.com